Effective Teaching and Learning

An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically.

The teaching of mathematics is complex. It requires teachers to have a deep understanding of the mathematical knowledge that they are expected to teach (Ball, Thames, and Phelps 2008) and a clear view of how student learning of that mathematics develops and progresses across grades (Daro, Mosher, and Corcoran 2011; Sztajn et al. 2012). It also requires teachers to be skilled at teaching in ways that are effective in developing mathematics learning for all students. This section presents, describes, and illustrates a set of eight research-informed teaching practices that support the mathematics learning of all students. Before turning to these teaching practices, however, we must be clear about the mathematics learning such teaching must inspire and develop and the inextricable connection between teaching and learning.

The learning of mathematics has been defined to include the development of five interrelated strands that, together, constitute mathematical proficiency (National Research Council 2001):

1. Conceptual understanding
2. Procedural fluency
3. Strategic competence
4. Adaptive reasoning
5. Productive disposition

Conceptual understanding (i.e., the comprehension and connection of concepts, operations, and relations) establishes the foundation, and is necessary, for developing procedural fluency (i.e., the meaningful and flexible use of procedures to solve problems).

Strategic competence (i.e., the ability to formulate, represent, and solve mathematical problems) and adaptive reasoning (i.e., the capacity to think logically and to justify one's thinking) reflect the need for students to develop mathematical ways of thinking as a basis for solving mathematics problems that they may encounter in real life, as well as within mathematics and other disciplines. These ways of thinking are variously described as "processes" (in NCTM's [2000] Process Standards), "reasoning habits" (NCTM 2009), or "mathematical practices" (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center and CCSSO] 2010). In this publication, in alignment with the Common

Core State Standards for Mathematics (CCSSM), we refer to them as "mathematical practices," which represent what students are doing as they learn mathematics (see fig. 1).

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Fig. 1. Standards for Mathematical Practice (NGO Center and CCSSO 2010, pp. 6–8)

The fifth strand identified on the preceding page, productive disposition, is "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (National Research Council 2001, p. 131). Students need to recognize the value of studying mathematics and believe that they are capable of learning mathematics through resolve and effort (Schunk and Richardson 2011). This conviction increases students' motivation and willingness to persevere in solving challenging problems in the short term and continuing their study of mathematics in the long term. Interest and curiosity evoked throughout the study of mathematics can spark a lifetime of positive attitudes toward the subject.

Student learning of mathematics "depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (Ball and Forzani 2011, p. 17). Ball and other researchers (e.g., Ball et al. 2009; Grossman, Hammerness, and McDonald 2009; Lampert 2010; McDonald, Kazemi, and Kavanagh 2013) argue that the profession of teaching needs to identify and work together toward the implementation of a common set of high-leverage practices that underlie effective teaching. By "high-leverage practices," they mean "those practices at the heart of the work of teaching that are most likely to affect student learning" (Ball and Forzani 2010, p. 45).

Although effective teaching of mathematics may have similarities with productive teaching in other disciplines (Duit and Treagust 2003; Hlas and Hlas 2012), each discipline requires focused attention on those teaching practices that are most effective in supporting student learning specific to the discipline (Hill et al. 2008; Hill, Rowan, and Ball 2005). Research from both cognitive science (Mayer 2002; Bransford, Brown, and Cocking 2000; National

Research Council 2012a) and mathematics education (Donovan and Bransford 2005; Lester 2007) supports the characterization of mathematics learning as an active process, in which each student builds his or her own mathematical knowledge from personal experiences, coupled with feedback from peers, teachers and other adults, and themselves. This research has identified a number of principles of learning that provide the foundation for effective mathematics teaching. Specifically, learners should have experiences that enable them to—

- engage with challenging tasks that involve active meaning making and support meaningful learning;
- connect new learning with prior knowledge and informal reasoning and, in the process, address preconceptions and misconceptions;
- acquire conceptual knowledge as well as procedural knowledge, so that they can meaningfully organize their knowledge, acquire new knowledge, and transfer and apply knowledge to new situations;
- construct knowledge socially, through discourse, activity, and interaction related to meaningful problems;
- receive descriptive and timely feedback so that they can reflect on and revise their work, thinking, and understandings; and
- develop metacognitive awareness of themselves as learners, thinkers, and problem solvers, and learn to monitor their learning and performance.

Mathematics Teaching Practices

Eight Mathematics Teaching Practices provide a framework for strengthening the teaching and learning of mathematics. This research-informed framework of teaching and learning reflects the learning principles listed above, as well as other knowledge of mathematics teaching that has accumulated over the last two decades. The list on the following page identifies these eight Mathematics Teaching Practices, which represent a core set of high-leverage practices and essential teaching skills necessary to promote deep learning of mathematics.

Obstacles

Dominant cultural beliefs about the teaching and learning of mathematics continue to be obstacles to consistent implementation of effective teaching and learning in mathematics classrooms (Handal 2003; Philipp 2007). Many parents and educators believe that students should be taught as they were taught, through memorizing facts, formulas, and procedures and then practicing skills over and over again (e.g., Sam and Ernest 2000). This view perpetuates the traditional lesson paradigm that features review, demonstration, and practice and is still pervasive in many classrooms (Banilower et al. 2006; Weiss and Pasley 2004). Teachers, as well

Mathematics Teaching Practices
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

as parents, are often not convinced that straying from these established beliefs and practices will be more effective for student learning (Barkatsas and Malone 2005; Wilken 2008).

In sharp contrast to this view is the belief that mathematics lessons should be centered on engaging students in solving and discussing tasks that promote reasoning and problem solving (NCTM 2009; National Research Council 2012a). Teachers who hold this belief plan lessons to prompt student interactions and discourse, with the goal of helping students make sense of mathematical concepts and procedures. However, the lack of agreement about what constitutes effective mathematics teaching constrains schools and school systems from establishing coherent expectations for high-quality, productive teaching of mathematics (Ball and Forzani 2011).

Teachers' beliefs influence the decisions that they make about the manner in which they teach mathematics, as indicated in the table at the right. Students' beliefs influence their

Beliefs about teaching and learning mathematics	
Unproductive beliefs	**Productive beliefs**
Mathematics learning should focus on practicing procedures and memorizing basic number combinations.	Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse.
Students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.	All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.
Students can learn to apply mathematics only after they have mastered the basic skills.	Students can learn mathematics through exploring and solving contextual and mathematical problems.
The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.	The role of the teacher is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics.
The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.	The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.
An effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.	An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.

perception of what it means to learn mathematics and their dispositions toward the subject. As the table summarizes, the impact of these beliefs on the teaching and learning of mathematics may be unproductive or productive. It is important to note that these beliefs should not be viewed as good or bad. Instead, beliefs should be understood as unproductive when they hinder the implementation of effective instructional practice or limit student access to important mathematics content and practices.

Overcoming the obstacles

Teaching mathematics requires specialized expertise and professional knowledge that includes not only knowing mathematics but knowing it in ways that make it useful for the work of teaching (Ball and Forzani 2010; Ball, Thames, and Phelps 2008). Mathematics teaching